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On the multi trace superpotential and corresponding matrix model

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Abstract. We study $\mathcal{N}=1$ supersymmetric $U(N)$ gauge theory coupled to an adjoint scalar superfield with a cubic superpotential containing a multi trace term. We show that the field theory results can be reproduced from a matrix model whose potential is given in terms of a linearized potential obtained from the gauge theory superpotential by adding some auxiliary non-dynamical field. Once we get the effective action from this matrix model we could integrate out the auxiliary field getting the correct field theory results.

1 Introduction

Recently it has been proposed [1] that the exact superpotential and gauge coupling for a wide class of $\mathcal{N}=1$ supersymmetric gauge theories can be obtained using perturbative computations in a *related* matrix model. Given an $\mathcal{N}=1$ SYM theory the potential of the corresponding matrix model is given in terms of the gauge theory superpotential. Even more interestingly, the non-perturbative results of the gauge theory can be obtained from just planar diagrams of the matrix model without taking any large N limit on the gauge theory side. This conjecture is based on earlier works [2–6] and has recently been verified perturbatively using the superspace formalism [7] or anomalies [8,9]. Further developments can be found in [10–14].

To fix our notation, let us consider $\mathcal{N}=1$ $U(N)$ supersymmetric gauge theory coupled to a chiral superfield, ϕ, in the adjoint representation with the following superpotential:

$$W(\phi)=\sum_{k=1}^{n+1}\frac{g_k}{k}\,\mathrm{Tr}\left(\phi^k\right) \tag{1}$$

for some n. To get a supersymmetric vacuum one needs to impose D- and F-term conditions. Taking ϕ to be diagonal would satisfy the D-term and for the F-term we need to set $W'(\phi)=0$. This equation, in general, has n distinct roots a_i and thus $W'(x)=g_{n+1}\prod_{i=1}^{n}(x-a_i)$. Therefore by taking ϕ to have eigenvalues a_i with multiplicity N_i, the gauge symmetry $U(N)$ is broken down to $\prod_{i=1}^{n}U(N_i)$ with $\sum_{i=1}^{n}N_i=N$.

If the roots a_i are all distinct, the chiral superfields are all massive and can then be integrated out getting an effective action for the low-energy theory. The chiral part of the low-energy effective Lagrangian may be written as [9]

$$L_{\mathrm{eff}}=\int \mathrm{d}^2\theta\, W_{\mathrm{eff}}+\mathrm{c.c.}\,,\quad W_{\mathrm{eff}}=f(S_k,g_k)+\sum_{i,j}\tau_{ij}\omega_{\alpha i}\omega_j^{\alpha}\,, \tag{2}$$

where $S_k=-\frac{1}{32\pi^2}\mathrm{Tr}\,W_{\alpha i}W^{\alpha i}$ and $\omega_{\alpha i}=\frac{1}{4\pi}\mathrm{Tr}\,W_{\alpha i}$ with $W^{\alpha i}$ being the gauge superfields of $U(N_i)$ gauge group.

The main point in the Dijkgraaf–Vafa proposal is that the chiral part of the effective action can be given by a holomorphic function $\mathcal{F}_G(S_k)$, such that

$$W_{\mathrm{eff}}=\sum_{i=1}^{n}N_i\frac{\partial\mathcal{F}_G}{\partial S_i}+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\mathcal{F}_G}{\partial S_i\partial S_j}\omega_{\alpha i}\omega_j^{\alpha}\,. \tag{3}$$

Now what is left to be determined is the function $\mathcal{F}_G$. In fact it is the goal of the Dijkgraaf–Vafa proposal to identify $\mathcal{F}_G$ as the free energy of an auxiliary non-supersymmetric matrix model whose potential is the same function W which is the superpotential of the four dimensional supersymmetric gauge theory. The matrix model free energy is given by

$$\mathrm{e}^{\frac{1}{g_s^2}\mathcal{F}_0}=\frac{1}{\mathrm{Vol}(U(M))}\int D\phi\,\mathrm{e}^{\left(-\frac{1}{g_s}W(\phi)\right)}\,, \tag{4}$$

where ϕ is an $M\times M$ matrix belonging to $U(M)$. For the model we are considering one needs to take ϕ in such a way that the $U(M)$ symmetry is also broken down to $\prod_{i=1}^{n}U(M_i)$ with $\sum_{i=1}^{n}M_i=M$. Moreover one should also identify $S_i=g_sM_i$. Taking the large M limit one can compute $\mathcal{F}_0$ order by order using only planar diagrams in the matrix model perturbation theory. Now the prescription [7] is that, for example, the lth instanton contribution to the effective action can be reproduced from a perturbative contribution with l loops in the auxiliary matrix model. In fact, having the matrix model free energy the

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effective superpotential is obtained:

$$W_{\text{eff}} = \sum_{i=1}^{n} \left(N_i \frac{\partial \mathcal{F}_0}{\partial S_i} - 2\pi i \tau_0 S_i \right) , \tag{5}$$

where τ_0 is the bare coupling of the theory.

By now there are a huge number of papers devoted to this proposal where only superpotentials with single trace operators have been studied. Recently a superpotential containing multi trace operators has also been considered in [15] where the authors showed that taking naively W with multi trace as the potential of the matrix model would lead to an *incorrect* matrix model. By "incorrect" they mean that one cannot reproduce the corresponding gauge theory results, though the obtained matrix model could be an auxiliary matrix model of some gauge theory which, of course, is not what we started with. More precisely it has been shown that although the diagrams surviving the large M limit of the matrix model with a multi trace potential are exactly the graphs that contribute to the effective action of the field theory with the multi trace tree level superpotential, one cannot compute the effective superpotential of the field theory by taking a derivative $\frac{\partial \mathcal{F}_0}{\partial S}$.

This problem can, of course, be solved [15] using the linearized superpotential in the matrix model. In fact, starting with a multi trace operator in the superpotential one can linearize it using some non-dynamical background fields A_i. Then the potential would contain only single trace operators with A_i dependent coefficients. Once we find W_{eff} from the matrix model, we can integrate out the A_i fields getting the *correct* gauge theory result.

It is the aim of this article to further study a superpotential with multi trace operators. In particular we would like to see whether the results of [15] can also be applied to the cases where the gauge group is broken too. We note that in [15] the authors have only considered the theory with a quadratic multi trace superpotential for a situation where the gauge symmetry remains unbroken. In fact, the main result of this paper is to show that the procedure works in the case with broken gauge symmetry as well.

For our purpose we shall study $\mathcal{N} = 1$ $U(N)$ SYM theory coupled to an adjoint scalar superfield with the cubic superpotential given by[1]

$$W_{\text{tree}} = \frac{1}{3} \operatorname{Tr}\left(\phi^3\right) + \frac{1}{2} m \operatorname{Tr}\left(\phi^2\right) + \lambda \operatorname{Tr}(\phi) + \frac{1}{2} g \operatorname{Tr}(\phi) \operatorname{Tr}\left(\phi^2\right) . \tag{6}$$

We will see that, using the linearized form of the superpotential, one can reproduce the gauge theory results for the cases with and without gauge symmetry breaking.

The organization of this paper is as follows. In Sect. 2 we will review $\mathcal{N} = 1$ $U(N)$ SYM theory with cubic single trace superpotential. In Sect. 3 we will study the same theory with a multi trace term added to the superpotential. Regarding the fact that this model can be thought of as a deformation of $\mathcal{N} = 2$ theory we will find the effective superpotential using the factorization of the Seiberg–Witten curve. In Sect. 4 we will reproduce the same field theory results using the linearized superpotential. In Sect. 5 we will see how the corresponding matrix model can be treated. The last section is devoted to our conclusions.

[1] $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory with cubic single trace superpotential has been extensively studied, for example, in [16–20]

2 Single trace superpotential

In this section we shall review the $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory coupled to an adjoint scalar hypermultiplet with cubic superpotential containing only single trace operators,

$$W_{\text{tree}} = \frac{1}{3} \operatorname{Tr}\left(\phi^3\right) + \frac{1}{2} m \operatorname{Tr}\left(\phi^2\right) + \lambda \operatorname{Tr}(\phi) . \tag{7}$$

Taking ϕ diagonal one just needs to set $W'(\phi) = 0$ to get the supersymmetric vacuum, and therefore the derivative of the superpotential can be recast to

$$W'(x) = (x - a_1)(x - a_2) , \quad a_{1,2} = -\frac{m}{2} \pm \frac{1}{2}\sqrt{m^2 - 4\lambda} . \tag{8}$$

In general we can take ϕ to have eigenvalues a_1 or a_2 with multiplicity N_1 and N_2, respectively. This will break gauge symmetry to $U(N_1) \times U(N_2)$ with $N_1 + N_2 = N$. Of course as a special case one can, for example, take $N_2 = 0$, which corresponds to the supersymmetric vacuum without gauge symmetry breaking. In the following we shall consider both cases.

2.1 Unbroken gauge symmetry

In this subsection we will review how the exact superpotential can be obtained for the case where the gauge symmetry is not broken, using the factorization of the Seiberg–Witten curve. In fact, the model we are interested in can be obtained from $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory perturbed by a general tree level superpotential given by

$$W_{\text{tree}} = \sum_{k=1}^{n+1} \frac{1}{k} g_k \operatorname{Tr}\left(\phi^k\right) . \tag{9}$$

A generic point in the moduli space of the $U(N)$ $\mathcal{N} = 2$ theory will be lifted by adding such a superpotential. The points which are not lifted are precisely those where at least $N - n$ mutually local monopoles become massless. These considerations are equivalent to the requirement that the corresponding Seiberg–Witten curve has the factorization [3]

$$P_N^2(x, u) - 4\Lambda^{2N} = H_{N-n}^2(x) F_{2n}(x), \tag{10}$$

where $P_N(x, u)$ is an order N polynomial in x with coefficients determined by the VEVs of u_k. Λ is an ultraviolet cut-off, H and F are order $N - n$ and $2n$ polynomials in x, respectively.

The $N-n$ double roots place $N-n$ conditions on the original variables u_k. We can parameterize all the $\langle u_k\rangle$ by n independent variables α_j. In other words, the α_j then correspond to massless fields in the low-energy effective theory. If we know the exact effective action for these fields, to find the vacua, we simply minimize S_{eff}. Furthermore, substituting $\langle u_k\rangle$ back into the effective action gives the action for the vacua.

In general the factorization problem is hard to solve, but for the confining vacuum where all $N-1$ monopoles have condensed, there is a general solution given by Chebyshev polynomials.[2] In our case, we have the solution

$$\langle u_p\rangle = \frac{N}{p}\sum_{q=0}^{[p/2]} C_p^{2q} C_{2q}^{q} \Lambda^{2q} z^{p-2q},$$
$$C_n^p := \binom{n}{p} = \frac{n!}{p!(n-p)!}, \tag{11}$$

where $z=\frac{\langle u_1\rangle}{N}$. We note, however, that the above procedure is not the best one for comparison with the matrix model result because there is no gluino condensate, S. To compare the results, we need to "integrate in" [22] the glueball superfield as in [17].

In the model we are considering the exact superpotential is found to be

$$W_{\text{exact}} = \langle u_3\rangle + m\langle u_2\rangle + \lambda\langle u_1\rangle, \tag{12}$$

where

$$\langle u_1\rangle = Nz, \quad \langle u_2\rangle = \frac{N}{2}\left(z^2+2\Lambda^2\right),$$
$$\langle u_3\rangle = \frac{N}{3}\left(z^3+6\Lambda^2 z\right). \tag{13}$$

One can now "integrate in" the glueball superfield, S, getting the effective superpotential as follows:

$$W_{\text{eff}} = -NS\left(\log\left(\frac{S}{\Delta\Lambda^2}\right)-1\right) \tag{14}$$
$$-\frac{2N}{3}\frac{S^2}{\Delta^3}\left(3+16\frac{S}{\Delta^3}+140\frac{S^2}{\Delta^6}+512\frac{S^3}{\Delta^9}\right),$$

which is the exact effective action up to 5 instantons. Here $\Delta=\sqrt{m^2-4\lambda}$.

Using the Dijkgraaf–Vafa proposal one will be able to reproduce this result using a non-supersymmetric matrix model with the potential given by (7). Since we are interested in the case where the gauge symmetry is not broken one considers the expansion around a classical solution as the following:

$$\phi = a_1 1_{M\times M} + \varphi, \tag{15}$$

and therefore the potential of the matrix model reads

$$W(\varphi) = W(a_1) + \frac{1}{3}\operatorname{Tr}\left(\varphi^3\right) + \frac{1}{2}\Delta\operatorname{Tr}\left(\varphi^2\right), \tag{16}$$

[2] This was worked out first by Douglas and Shenker [21]

where $\Delta = a_1 - a_2$. Here ϕ is an $M\times M$ matrix belonging to the $U(M)$ group. One can now write the Feynman rules and thereby evaluate the matrix model free energy order by order using (4). Here we shall take a limit in which M is large and keep the 't Hooft coupling $S=g_sM$ fixed, and thus only planar diagrams would contribute. In this limit the free energy is found to be [16]

$$\mathcal{F}_0(S) = \frac{1}{2}S^2\log\left(\frac{S}{\Delta^3}\right) - S^2\log\left(\frac{\Lambda}{\Delta}\right)$$
$$+\frac{2}{3}\frac{S^3}{\Delta^3}\left(1+4\frac{S}{\Delta^3}+28\frac{S^2}{\Delta^6}+\ldots\right) \tag{17}$$

up to 4 loops. Using this expression the exact superpotential is given by (see also [3])

$$W_{\text{exact}} = -NS\left(\log\left(\frac{S}{\Delta\Lambda^2}\right)-1\right) \tag{18}$$
$$-\frac{2N}{3}\frac{S^2}{\Delta^3}\left(3+16\frac{S}{\Delta^3}+140\frac{S^2}{\Delta^6}+\ldots\right),$$

which is in exact agreement with the field theory computation (14). As we see, the lth loop contribution to the matrix model free energy is the same as the l instantons contribution to the effective action.

2.2 Broken gauge symmetry

In this subsection we shall review the case where the gauge symmetry is broken to two parts. In other words we consider a matrix model where the $U(M)$ group is broken down to $U(M_1)\times U(M_2)$. To get such a matrix model we take

$$\phi = \begin{pmatrix} a_1 1_{M_1\times M_1} & 0 \\ 0 & a_2 1_{M_2\times M_2}\end{pmatrix} + \begin{pmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{pmatrix}; \tag{19}$$

here $M_1+M_2=M$. Moreover we will consider the large M_1 and M_2 limit while keeping $S_1=g_sM_1$ and $S_2=g_sM_2$ fixed. The free energy of the corresponding matrix model around this vacuum, up to 4 loops, is given by [16]

$$\mathcal{F}_0(S_1,S_2) = -\frac{1}{2}\sum {S_i}^2\log\left(\frac{S_i}{\Delta^3}\right)$$
$$+(S_1+S_2)^2\log\left(\frac{\Lambda}{\Delta^3}\right)$$
$$+\frac{1}{3\Delta^3}\left(2S_1^3-15S_1^2S_2+15S_1S_2^2-2S_2^3\right)$$
$$+\frac{1}{3\Delta^6}\left(8S_1^4-91S_1^3S_2+177S_1^2S_2^2-91S_1S_2^3-8S_2^4\right)$$
$$+\frac{1}{3\Delta^9}\left(56S_1^5-871S_1^4S_2+2636S_1^3S_2^2-2636S_1^2S_2^3\right.$$
$$\left.+871S_1S_2^4-56S_2^5\right). \tag{20}$$

Having the matrix model free energy the effective superpotential for the case where the gauge symmetry is broken as $U(N)\to U(N_1)\times U(N_2)$ can be found using (5).

This effective superpotential should be compared with that obtained from a gauge theory computation. The gauge theory result may be found using the factorization of the Seiberg–Witten curve, though, in general, the factorization procedure is difficult to do. Nevertheless for a special case this can easily be worked out. For example consider the SYM theory with gauge group $U(3N)$ broken down to $U(2N)\times U(N)$. Actually the analysis of this theory is equivalent to SYM theory with gauge group $U(3)$ broken down to $U(2)\times U(1)$ where the effective superpotential turned out to be [23] (see also [26])

$$W_{\rm eff}=u_3+mu_2+\lambda u_1\pm 2\Lambda^3\,. \tag{21}$$

Of course this is not a suitable form for comparison with the matrix model result. Actually to compare these two results one can, for example, integrate out the S_1 and S_2 fields from the effective superpotential obtained from the matrix model. Doing so, one can see that the matrix model reproduces the correct result (21) order by order [3].

3 Multi trace superpotential

In this section we will study $\mathcal{N}=1\;U(N)$ SYM theory coupled to an adjoint scalar superfield with a superpotential containing a multi trace operator

$$W_{\rm tree}=\frac{1}{3}\,{\rm Tr}\left(\phi^3\right)+\frac{1}{2}\,m\,{\rm Tr}\left(\phi^2\right)+\lambda\,{\rm Tr}\left(\phi\right)+\frac{1}{2}\,g\,{\rm Tr}\left(\phi\right){\rm Tr}\left(\phi^2\right)\,. \tag{22}$$

To get the supersymmetric vacuum one needs to impose F- and D-terms conditions. Taking a diagonal ϕ would satisfy the D-term condition and for the F-term we need to solve $W'_{\rm tree}(\phi)=0$. This equation has two solutions, $b_{1,2}$, and therefore in general ϕ can be taken to have eigenvalues b_i with multiplicity N_i. This will break the gauge symmetry down to $U(N_1)\times U(N_2)$ with $N_1+N_2=N$.

To find the eigenvalues b_i we note that the adjoint scalar has been taken as $\phi={\rm diag}(b_1 1_{N_1\times N_1},b_2 1_{N_2\times N_2})$, and thus the superpotential is given by

$$\begin{aligned}W_{\rm tree}=&\frac{1}{3}\left(N_1b_1^3+N_2b_2^3\right)+\frac{m}{2}\left(N_1b_1^2+N_2b_2^2\right)\\&+\lambda\left(N_1b_1+N_2b_2\right)\\&+\frac{g}{2}\left(N_1b_1+N_2b_2\right)\left(N_1b_1^2+N_2b_2^2\right)\,,\end{aligned} \tag{23}$$

and therefore the F-term condition reads

$$\begin{aligned}&\lambda+mb_1+b_1^2+\frac{g}{2}\left(N_1b_1^2+N_2b_2^2\right)+gb_1(N_1b_1+N_2b_2)=0\,,\\&\lambda+mb_2+b_2^2+\frac{g}{2}\left(N_1b_1^2+N_2b_2^2\right)+gb_2(N_1b_1+N_2b_2)=0\,.\end{aligned} \tag{24}$$

One can now solve these equations to find b_1 and b_2. The solution is

$$b_1=-\frac{m}{1+N_1g}-\frac{1+N_2g}{1+N_1g}\,b_2, \tag{25}$$

with b_2 satisfying

$$b_2^2+\tilde{m}b_2+\tilde{\lambda}=0\,, \tag{26}$$

where

$$\begin{aligned}\tilde{m}&=\frac{\left(1+2N_1+N_1N_2g^2\right)m}{(1+(N_1+N_2)g)\left(3+N_1N_2g^2\right)-1}\,,\\\tilde{\lambda}&=\frac{2\lambda\left(1+N_1g\right)^2+m^2N_1g}{(1+(N_1+N_2)g)\left(3+N_1N_2g^2\right)-1}\,.\end{aligned} \tag{27}$$

Thus in general one can write $W'(x)=(x-b_1)(x-b_2)$.

3.1 Unbroken gauge symmetry

By making use of the fact that this model can be obtained from a deformation of $\mathcal{N}=2\;U(N)$ SYM theory by adding the superpotential (22), the effective superpotential can be obtained from the factorization of the Seiberg–Witten curve. In fact, since the gauge symmetry is not broken the factorization, for the confining vacuum where all $N-1$ monopoles have condensed, can be obtained using Chebyshev polynomials. Indeed the solution is the same as (11). Therefore the effective superpotential reads

$$\begin{aligned}W_{\rm exact}=&N\left(\lambda+2\Lambda^2+gN\Lambda^2\right)z+\frac{mN}{2}\left(z^2+2\Lambda^2\right)\\&+\frac{N}{3}\left(1+\frac{3gN}{2}\right)z^3\,.\end{aligned} \tag{28}$$

Setting $B:=\Lambda^2$ one can integrate in the glueball field S as follows. First we find B in terms of S from the equation

$$NS=B\frac{\partial W_{\rm exact}}{\partial B}=NB\left(m+2z+gNz\right)\,, \tag{29}$$

which can be interpreted as the Konishi anomaly [9, 24].[3]

Then we find z by solving

$$\begin{aligned}0&=\frac{\partial W_{\rm exact}}{\partial z}\\&=N\left(\left(1+\frac{3gN}{2}\right)z^2+mz+\lambda+2B+gNB\right)\,.\end{aligned} \tag{30}$$

The effective action for the glueball superfield S can be written as

$$W_{\rm eff}(S,g,\Lambda)=-S\log\left(\frac{B}{\Lambda^2}\right)^N+W_{\rm exact}(S,g). \tag{31}$$

To write the effective superpotential explicitly let us, for simplicity, set $\lambda=0$. In this case one finds the following solutions for z and B in power series of S up to order $\mathcal{O}(S^6)$:

$$B=\frac{S}{m}+\frac{(2+gN)^2S^2}{m^4}+\frac{(10+7gN)(2+gN)^3S^3}{2m^7}$$

[3] This might also be related to the non-perturbative relation studied in, for example, [25]

$$+\frac{\left(32+46gN+17g^2N^2\right)(2+gN)^4S^4}{m^{10}}$$
$$+\frac{\left(1848+4044gN+3018g^2N^2+769g^3N^3\right)}{8m^{13}}$$
$$\times(2+gN)^5S^5$$
$$z=-\frac{(2+gN)^2S^2}{m^2}-\frac{(6+5gN)(2+gN)^2S^2}{2m^5}$$
$$-\frac{\left(16+16gN+11g^2N^2\right)(2+gN)^3S^3}{m^8}$$
$$-\frac{5(6+5gN)\left(28+44gN+19g^2N^2\right)(2+gN)^4S^4}{8m^{11}}$$
$$-\frac{\left(3072+9768gN+11940g^2N^2+6654g^3N^3+1427g^4N^4\right)}{4m^{14}}$$
$$\times(2+gN)^5S^5. \tag{32}$$

Plugging these solutions into (31) one gets the effective superpotential as follows:

$$W_{\text{eff}}=-NS\left(\log\left(\frac{S}{\Lambda^2}\right)-1\right)-\frac{N(2+gN)^2S^2}{2m^3}$$
$$-\frac{N(4+3gN)(2+gN)^3S^3}{3m^6}$$
$$-\frac{N\left(140+212gN+83g^2N^2\right)(2+gN)^4S^4}{24m^9}$$
$$-\frac{N\left(128+292gN+228g^2N^2+61g^3N^3\right)}{4m^{12}}$$
$$\times(2+gN)^5S^5\,. \tag{33}$$

As a check for this expression we note that setting $g=0$ we will get the same result as in the single trace case.

3.2 Broken gauge symmetry

In this case, to get a closed form for the effective superpotential we will consider the case where the gauge symmetry $U(3N)$ is broken down to $U(2N)\times U(N)$. Essentially this is equivalent to the case with $U(3)\to U(2)\times U(1)$ symmetry breaking. To find the effective superpotential one can use the factorization of the Seiberg–Witten curve. The corresponding Seiberg–Witten curve for $U(3)$ theory is given by [27, 28]

$$y^2=\left(x^3-s_1x^2-s_2x-s_3\right)^2-4\Lambda^6, \tag{34}$$

and the factorization we are interested in is

$$\left(x^3-s_1x^2-s_2x-s_3\right)^2-4\Lambda^6=H_1(x)^2F_4(x)\,. \tag{35}$$

Following [3] one finds

$$s_i=s_i^{\text{class}}\pm 2\Lambda^3\delta_{i,3}\,, \tag{36}$$

where

$$s_1^{\text{class}}=2b_1+b_2\,,\quad s_2^{\text{class}}=-2b_1b_2+b_1^2\,,\quad s_3^{\text{class}}=b_1^2b_2\,. \tag{37}$$

Therefore the effective superpotential reads

$$W_{\text{eff}}=u_3^{\text{class}}+mu_2^{\text{class}}+\lambda u_1^{\text{class}}+gu_1^{\text{class}}u_2^{\text{class}}\pm 2\Lambda^3\,, \tag{38}$$

where

$$u_1^{\text{class}}=2b_1+b_2\,,\quad u_2^{\text{class}}=b_1^2+\frac{1}{2}b_2^2\,,\quad u_3^{\text{class}}=\frac{2}{3}b_1^2+\frac{}{3}b_2^2\,. \tag{39}$$

4 Linearized superpotential

4.1 Field theory description

Following [15] one can recast the superpotential to the form with only single trace operators using auxiliary fields. In our case we need two fields A_1 and A_2 and the superpotential may be written as

$$W_{\text{tree}}=\frac{1}{3}\,\mathrm{Tr}\left(\phi^3\right)+\frac{1}{2}(m+gA_1)\,\mathrm{Tr}\left(\phi^2\right)$$
$$+(\lambda+gA_2)\,\mathrm{Tr}(\phi)-gA_1A_2\,. \tag{40}$$

Since A_1 and A_2 have no dynamics, one can integrate them out and refine the multi trace superpotential (22). These fields can be treated as constant background fields and therefore the theory can be solved using a single trace superpotential. This will generate an effective superpotential $W_{\text{eff}}^{\text{single}}(A_1,A_2,S)$ as a function of the A_i and the glueball superfield S. This function is the same as that in the model without multi trace but with A_i dependent couplings.

For example in the case where the gauge group is not broken the effective superpotential can be read from the single trace result (12)

$$W_{\text{exact}}^{\text{single}}(A_1,A_2)=\langle u_3\rangle+m'\langle u_2\rangle+\lambda'\langle u_1\rangle\,, \tag{41}$$

where $m'=m+gA_1$, $\lambda'=\lambda+gA_2$ and

$$\langle u_1\rangle=Nz\,,\quad \langle u_2\rangle=\frac{N}{2}\left(z^2+2\Lambda^2\right),$$
$$\langle u_3\rangle=\frac{N}{3}\left(z^3+6\Lambda^2z\right)\,. \tag{42}$$

In the same way as in the previous section one can proceed to “integrate in” the glueball superfield. To do this one sets $B:=\Lambda^2$ and uses the equation

$$NS=B\frac{\partial W_{\text{exact}}^{\text{single}}}{\partial B}=NB(m'+2z), \tag{43}$$

to solve for B in terms of S. One can also find z by solving

$$0=\frac{\partial W_{\text{exact}}^{\text{single}}}{\partial z}=N(z^2+m'z+\lambda'+2B)\,. \tag{44}$$

Now the effective superpotential for the glueball superfield can be written as

$$W_{\text{eff}}^{\text{single}}(A_1,A_2,S)=-S\log\left(\frac{B}{\Lambda^2}\right)^N+W_{\text{exact}}^{\text{single}}(A_1,A_2,S)\,. \tag{45}$$

In fact, using the result of the single trace model (14) we get

$$W_{\rm eff}^{\rm single}(A_1,A_2,S) = -NS\left(\log\left(\frac{S}{\Delta'\Lambda^2}\right)-1\right) \quad (46)$$
$$-\frac{2N}{3}\frac{S^2}{\Delta'^3}\left(3+16\frac{S}{\Delta'^3}+140\frac{S^2}{\Delta'^6}+512\frac{S^3}{\Delta'^9}\right),$$

where $\Delta'^2=(m+gA_1)^2-4(\lambda+gA_2)$. One should also add to this the $-gA_1A_2$ term, and the final answer for the superpotential is

$$W_{\rm eff}(A_1,A_2,S) = W_{\rm eff}^{\rm single}(A_1,A_2,S) - gA_1A_2. \quad (47)$$

To get the final result for the effective superpotential with multi trace operator we need to integrate out the A_i using their equations of motion

$$\frac{\partial W_{\rm eff}^{\rm single}(A_1,A_2,S)}{\partial A_1} - gA_2 = 0\,,$$
$$\frac{\partial W_{\rm eff}^{\rm single}(A_1,A_2,S)}{\partial A_2} - gA_1 = 0\,. \quad (48)$$

These equations can be solved to find the A_i in terms of the glueball superfield, and then plugging back the results into (47) one can obtain the effective superpotential for the theory with tree level superpotential (22). This should reproduce the field theory result of the multi trace superpotential (33). This can be seen as follows.

Suppose we have been able to solve (43) and (44) exactly. Then we would have the exact form of z and B as functions of S, A_1 and A_2:

$$B=B(A_1,A_2,S)\,,\quad z=z(A_1,A_2,S)\,. \quad (49)$$

Plugging these into the effective superpotential (47) one gets

$$W_{\rm eff}(A_1,A_2,S) = \frac{N}{3}\left(z^3+6\Lambda^2 z\right)$$
$$+\frac{N}{2}(m+gA_1)\left(z^2+2\Lambda^2\right)+(\lambda+gA_2)Nz$$
$$-S\log\left(\frac{B}{\Lambda^2}\right)^N - gA_1A_2. \quad (50)$$

Thus the equations of motion of the A_i read

$$\frac{\partial W_{\rm eff}}{\partial A_1}+\frac{\partial B}{\partial A_1}\frac{\partial W_{\rm eff}}{\partial B}+\frac{\partial z}{\partial A_1}\frac{\partial W_{\rm eff}}{\partial z}-gA_2=0\,,$$
$$\frac{\partial W_{\rm eff}}{\partial A_2}+\frac{\partial B}{\partial A_2}\frac{\partial W_{\rm eff}}{\partial B}+\frac{\partial z}{\partial A_2}\frac{\partial W_{\rm eff}}{\partial z}-gA_1=0\,, \quad (51)$$

which are

$$0=\frac{gN}{2}\left(z^2+2B\right)-gA_2$$
$$+N\frac{\partial B}{\partial A_1}\left(-\frac{S}{B}+(m+z+gA_1)\right)$$
$$+N\frac{\partial z}{\partial A_1}\left(z^2+(m+gA_1)z+\lambda+gA_2+B\right),$$
$$0=gNz-gA_1+N\frac{\partial B}{\partial A_2}\left(-\frac{S}{B}+(m+z+gA_1)\right)$$
$$+N\frac{\partial z}{\partial A_2}\left(z^2+(m+gA_1)z+\lambda+gA_2+B\right). \quad (52)$$

By making use of (43) and (44) we find

$$A_2=\frac{N}{2}\left(z^2+2B\right)\,,\quad A_1=Nz\,. \quad (53)$$

Now one has to plug these solutions into the effective superpotential to get the final result which is, of course, what we have found in the previous section, (33).

In the case where the gauge group is broken to two parts we can follow the same procedure. To be specific we consider $U(3)\to U(2)\times U(1)$ where we will be able to write a closed form for the exact superpotential. More precisely, using the field theory result in the single trace case the effective superpotential reads

$$W_{\rm eff}(A_1,A_2) = (\lambda+gA_2)u'^{\rm class}_1+(m+gA_1)u'^{\rm class}_2$$
$$+u'^{\rm class}_3\pm 2\Lambda^3-gA_1A_2\,, \quad (54)$$

where

$$u'^{\rm class}_1=2a'_1+a'_2\,,\quad u'^{\rm class}_2=a'^2_1+\frac{a'^2_2}{2}\,,$$
$$u'^{\rm class}_3=\frac{2a'^3_1}{3}+\frac{a'^3_2}{3}\pm 2\Lambda^3\,, \quad (55)$$

with $a'_{1,2}=-\frac{m'}{2}\pm\frac{1}{2}\sqrt{m'^2-4\lambda'}$. We should now show that upon integrating out the auxiliary fields A_1 and A_2 the obtained effective action is the same as that in the field theory computation with multi trace operator (38). To see this, we note that

$$\frac{\partial W_{\rm eff}}{\partial A_1}=g\left(u'^{\rm class}_2-A_2\right)$$
$$+\left((\lambda+gA_2)\frac{\partial u'^{\rm class}_1}{\partial A_1}+(m+gA_1)\frac{\partial u'^{\rm class}_2}{\partial A_1}+\frac{\partial u'^{\rm class}_3}{\partial A_1}\right)$$
$$=0,$$
$$\frac{\partial W_{\rm eff}}{\partial A_2}=g\left(u'^{\rm class}_1-A_1\right)$$
$$+\left((\lambda+gA_2)\frac{\partial u'^{\rm class}_1}{\partial A_2}+(m+gA_1)\frac{\partial u'^{\rm class}_2}{\partial A_2}+\frac{\partial u'^{\rm class}_3}{\partial A_2}\right)$$
$$=0, \quad (56)$$

which leads to the following solution for the A_i:

$$A_1=u'^{\rm class}_1\,,\quad A_2=u'^{\rm class}_2\,. \quad (57)$$

From these expressions one can find A_1 and A_2 and plug them into the effective superpotential (54). Doing so we will get the same result as (38).

4.2 Matrix model description

In this section we study the matrix model of the gauge theory with multi trace operators. As it was shown in [15] naively taking the W including a multi trace operator as the potential of the corresponding matrix model would lead to an incorrect result. And in fact we should work with the linearized form of the superpotential. Therefore we consider the $U(M)$ matrix model with the following cubic potential:

$$W_{\text{tree}} = \frac{1}{3}\operatorname{Tr}\left(\phi^3\right) + \frac{1}{2}m'\operatorname{Tr}\left(\phi^2\right) + \lambda'\operatorname{Tr}(\phi) - gA_1A_2 \,. \tag{58}$$

This can be thought of as a matrix model with the single trace potential while treating the A_i as constant background fields plus a shift of the form $-gA_1A_2$. For the single trace part, the potential has two critical points a_1' and a_2' such that

$$W'(x) = (x-a_1')(x-a_2')\,,\quad a_{1,2}' = -\frac{m'}{2} \pm \frac{1}{2}\sqrt{m'^2-4\lambda'}\,. \tag{59}$$

In the case where the gauge symmetry is not broken one can take the following small fluctuations:

$$\phi = a_1' 1_{M\times M} + \varphi\,, \tag{60}$$

and therefore the potential of the matrix model reads

$$W(\varphi) = W(a_1') + \frac{1}{3}\operatorname{Tr}\left(\varphi^3\right) + \frac{1}{2}\Delta'\operatorname{Tr}\left(\varphi^2\right)\,, \tag{61}$$

where $\Delta = a_1' - a_2'$. We can now write down the Feynman rules and thereby evaluate the free energy order by order. Here we shall also consider the large M limit while keeping $g_sM = S$ fixed. Thus only planar diagrams would contribute. Basically using the single trace result as that in Sect. 2 we find

$$\mathcal{F}_0^{\text{single}}(A_1,A_2,S) = -\frac{1}{2}S^2\log\left(\frac{S}{\Delta'^3}\right) + S^2\log\left(\frac{\Lambda}{\Delta'}\right) + \frac{2}{3}\frac{S^3}{\Delta'^3}\left(1+4\frac{S}{\Delta'^3}+28\frac{S^2}{\Delta'^6}\right) \tag{62}$$

up to 4 loops. Using this expression, the exact superpotential is given by

$$W_{\text{eff}}^{\text{single}}(A_1,A_2,S) = -NS\left(\log\left(\frac{S}{\Delta'\Lambda^2}\right)-1\right) - \frac{2N}{3}\frac{S^2}{\Delta'^3}\left(3+16\frac{S}{\Delta'^3}+140\frac{S^2}{\Delta'^6}\right). \tag{63}$$

Finally the effective superpotential for the multi trace model can be found by integrating out A_1 and A_2 from the total superpotential given by

$$W_{\text{eff}}(A_1,A_2,S) = W_{\text{eff}}^{\text{single}}(A_1,A_2,S) - gA_1A_2\,, \tag{64}$$

This, of course, is the same expression as (47) and thus would lead to the correct answer. Therefore we might conclude that the linearized superpotential would give a correct matrix model for an $\mathcal{N}=1$ gauge theory with a multi trace operator in the superpotential.

On the other hand for the case where the gauge group is broken, we consider the large M $U(M)$ matrix model and take the small fluctuations as follows:

$$\phi = \begin{pmatrix} a_1 1_{M_1\times M_1} & 0 \\ 0 & a_2 1_{M_2\times M_2}\end{pmatrix} + \begin{pmatrix}\varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22}\end{pmatrix}\,, \tag{65}$$

with $M_1 + M_2 = M$. Therefore the gauge symmetry is broken down to $U(M_1)\times U(M_2)$. We shall also consider the large M_1 and M_2 limit while keeping $S_1 = g_sM_1$ and $S_2 = g_sM_2$ fixed. Using the single trace result the matrix model action is found to be

$$\begin{aligned} W &= \frac{1}{2}\Delta'\left(\operatorname{Tr}\left(\varphi_{11}^2\right) - \operatorname{Tr}\left(\varphi_{22}^2\right)\right) \\ &+\frac{1}{3}\left(\operatorname{Tr}\left(\varphi_{11}^3\right)+\operatorname{Tr}\left(\varphi_{22}^3\right)\right) \\ &+\Delta'\left(\operatorname{Tr}(B_{21}C_{12}) - \operatorname{Tr}(B_{12}C_{21})\right) \\ &+\operatorname{Tr}(B_{21}\varphi_{11}C_{12} + C_{21}\varphi_{11}B_{12}) \\ &+\operatorname{Tr}(B_{12}\varphi_{22}C_{21} + C_{11}\varphi_{22}B_{21})\,. \end{aligned} \tag{66}$$

Therefore the matrix model free energy up to 4-loop reads

$$\begin{aligned} \mathcal{F}_0(A_1,A_2,S_1,S_2) &= -\frac{1}{2}\sum {S_i}^2\log\left(\frac{S_i}{\Delta'^3}\right) \\ &+(S_1+S_2)^2\log\left(\frac{\Lambda}{\Delta'^3}\right) \\ &+\frac{1}{3\Delta'^3}\left(2S_1^3 - 15S_1^2S_2 + 15S_1S_2^2 - 2S_2^3\right) \\ &+\frac{1}{3\Delta'^6}\left(8S_1^4 - 91S_1^3S_2 + 177S_1^2S_2^2 - 91S_1S_2^3 - 8S_2^4\right) \\ &+\frac{1}{3\Delta'^9}\left(56S_1^5 - 871S_1^4S_2 + 2636S_1^3S_2^2 - 2636S_1^2S_2^3 \right. \\ &\left. + 871S_1S_2^4 - 56S_2^5\right). \end{aligned} \tag{67}$$

Having the explicit expression for the matrix model free energy with symmetry breaking as $U(M)\to U(M_1)\times U(M_2)$, one can find the effective superpotential $W_{\text{eff}}^{\text{single}}(A_i,S_i)$ for the gauge theory where the gauge group is broken as $U(N)\to U(N_1)\times U(N_2)$ by making use of (5). Then the effective superpotential for the multi trace theory can be obtained by integrating out the auxiliary fields A_i from

$$W_{\text{eff}}(A_i,S_i) = W_{\text{eff}}^{\text{single}}(A_i,S_i) - gA_1A_2\,. \tag{68}$$

To check the result one might consider the model with $N_1 = 2$ and $N_2 = 1$ where the field theory result is known. Doing the same analysis as before one can see that this does give the correct answer.

5 Conclusions

In this paper we have studied $\mathcal{N}=1$ supersymmetric $U(N)$ gauge theory coupled to an adjoint scalar superfield with a cubic superpotential containing a multi trace operator.

Next we have looked for the corresponding matrix model in the context of the Dijkgraaf–Vafa proposal.

Following [15] we have considered a matrix model whose potential is given by the linearized form of the superpotential of the corresponding gauge theory using some auxiliary fields. In this way the problem can be recast in the form of the single trace case with, of course, coefficients which now depend on the auxiliary fields. Using this matrix model one can find the free energy and thereby the effective superpotential using the Dijkgraaf–Vafa proposal. At the end we should integrate out the auxiliary fields finding the final result of the exact superpotential for the theory with multi trace in the tree level superpotential. As it was noticed in [15] it is crucial when the auxiliary fields are integrated out.

In this paper we have only considered the multi trace operator with the form $\mathrm{Tr}(\phi)\,\mathrm{Tr}(\phi^2)$, while we could have also considered other multi trace operators like $(\mathrm{Tr}(\phi))^3$. In this paper we have studied two different models: one with gauge symmetry breaking and the other without gauge symmetry breaking. In both cases we have seen that the linearized matrix model does give the correct field theory result.

In fact, one of our motivations for doing this project was to find whether the Dijkgraaf–Vafa proposal can also be applied to an exceptional group. We note, however, that the tree level superpotential of a gauge theory with an exceptional group has usually multi trace operators. For example $\mathcal{N}=1$ supersymmetric gauge theory with gauge group G_2 can be obtained from $\mathcal{N}=2$ G_2 SYM theory by a tree level superpotential given by

$$W_{\mathrm{tree}} = \frac{m}{4}\,\mathrm{Tr}\left(\phi^2\right) + \frac{g}{6}\left(\mathrm{Tr}\left(\phi^6\right) - \frac{1}{16}\,\mathrm{Tr}\left(\phi^2\right)^3\right). \quad (69)$$

So the first step to study these theories is to increase our knowledge of the physics of multi trace operators. We hope to address this issue in the future.

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